1. (a) Express $\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x$ as an integral.

(1)

(b) Hence show that

$$\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \ln k$$

where k is a constant to be found.

(2)

(a)
$$\lim_{\delta x \to 0} \sum_{x=2.1}^{6.3} \frac{2}{x} \delta x = \int_{2.1}^{6.3} \frac{2}{x} \, dx \quad ①$$

(b)
$$\int_{2.1}^{6.3} \frac{2}{x} \, dx = \left[2\ln x \right]_{2.1}^{6.3}$$
$$= (2\ln 6.3) - (2\ln 2.1) \quad ①$$
$$= 2\ln \left(\frac{6.3}{2.1} \right)$$
$$= 2\ln 3$$
$$= \ln 3^2$$
$$= \ln 9 \quad ①$$
$$\therefore k = 9$$

2. **In this question you must show all stages of your working.**

Solutions relying on calculator technology are not acceptable.

Show that

$$\int_1^{e^2} x^3 \ln x \, dx = ae^8 + b$$

where a and b are rational constants to be found.

(5)

$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}\,dx$

integration by parts:

$\int u \, dv = uv - \int v \, du$

$dv = x^3 \Rightarrow v = \frac{1}{4}x^4$

choose $\ln x$ as u because it is much easier to differentiate than integrate.

$\int x^3 \, dx = \frac{1}{4}x^4$

$\int u \, dv = uv - \int v \, du$

$$\int_1^{e^2} x^3 \ln x \, dx = \left[\ln x \times \frac{1}{4}x^4\right]_1^{e^2} - \int_1^{e^2} \frac{1}{x} \times \frac{x^4}{4} \, dx \quad ①$$

①

$\frac{1}{x} \times \frac{x^4}{4} = \frac{x^3}{4}$

$$= \left[\frac{x^4}{4}\ln x - \frac{x^4}{16}\right]_1^{e^2} \quad ①$$

$$= \left(\frac{{e^2}^4}{4}\ln(e^2) - \frac{{e^2}^4}{16}\right) - \left(\frac{1^4}{4}\ln 1 - \frac{1^4}{16}\right) \quad ①$$

$\ln e^2 = 2$ $\qquad$ $\ln 1 = 0$

$$= \left(\frac{2e^8}{4} - \frac{e^8}{16}\right) - \left(-\frac{1}{16}\right)$$

$$= \frac{7}{16}e^8 + \frac{1}{16} \quad ①$$

3. The table below shows corresponding values of x and y for $y = \log_3 2x$

The values of y are given to 2 decimal places as appropriate.

$n = 1.5$

x	3	4.5	6	7.5	9
y	1.63	2	2.26	2.46	2.63

(a) Using the trapezium rule with all the values of y in the table, find an estimate for

$$\int_3^9 \log_3 2x \, dx$$

(3)

Using your answer to part (a) and making your method clear, estimate

(b) (i) $\int_3^9 \log_3 (2x)^{10} \, dx$

(ii) $\int_3^9 \log_3 18x \, dx$

(3)

a) $\int_3^9 \log_3 2x \, dx \approx \frac{1.5}{2}\left(1.63 + 2.63 + 2(2 + 2.26 + 2.46)\right)$ (1) (1)

$= \frac{531}{40}$ (1) $= 13.3$

b) i) $\int_3^9 \log_3 (2x)^{10} \, dx = 10\int_3^9 \log_3 (2x) \, dx$

$= 10 \times 13.3$

$= 133$ (1)

(ii) $\int_3^9 \log_3 18x \, dx = \int_3^9 \log_3 9 \times 2x \, dx$

$= \int_3^9 \log_3 9 \, dx + \int_3^9 \log_3 2x \, dx$ 1

$= \int_3^9 2 \, dx + 13.275$

$= \left[2x \right]_3^9 + 13.275$

$= (2 \times 9) - (2 \times 3) + 13.3$

$= 18 - 6 + 13.3$

$= 25.3$ 1

4.

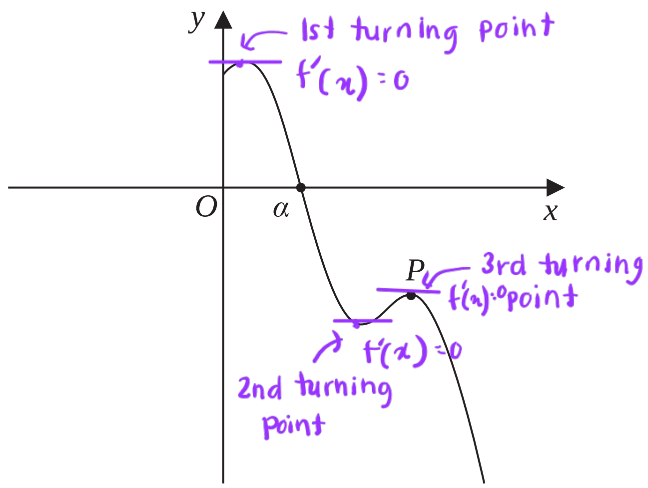

Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = f(x)$ where

$$f(x) = 8\sin\left(\frac{1}{2}x\right) - 3x + 9 \qquad x > 0$$

and x is measured in radians.

The point P, shown in Figure 2, is a local maximum point on the curve.

Using calculus and the sketch in Figure 2,

(a) find the x coordinate of P, giving your answer to 3 significant figures.

(4)

The curve crosses the x-axis at $x = \alpha$, as shown in Figure 2.

Given that, to 3 decimal places, $f(4) = 4.274$ and $f(5) = -1.212$

(b) explain why α must lie in the interval $[4, 5]$

(1)

(c) Taking $x_0 = 5$ as a first approximation to α, apply the Newton-Raphson method once to $f(x)$ to obtain a second approximation to α.

Show your method and give your answer to 3 significant figures.

(2)

a) $f(x) = 8\sin\left(\frac{1}{2}x\right) - 3x + 9$

$f'(x) = 8 \times \frac{1}{2}\cos\left(\frac{1}{2}x\right) - 3$ 1

$= 4\cos\left(\frac{1}{2}x\right) - 3$ 1

at P, $f'(x) = 0$

$4\cos\left(\frac{1}{2}x\right) = 3$ ①

$\cos\left(\frac{1}{2}x\right) = \frac{3}{4}$

$2\pi - 0.7227$

$\frac{1}{2}x = \cos^{-1}\left(\frac{3}{4}\right)$

$2\pi + 0.7227$

$\frac{1}{2}x = 0.7227, \ 5.560, \ 7.006$

$x = 1.445\ldots, \ 11.120\ldots, \ 14.011\ldots$

$\therefore P_x = 14.0 \ (3\text{s.f.})$ ① 3rd turning point

b) $f(4) = 4.274 > 0$ and $f(5) = -1.212 < 0$ ①

There is a change of sign between $f(4)$ and $f(5)$.

The function is continuous in interval $[4, 5]$

$\therefore 4 < \alpha < 5$

c) Newton Raphson

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_0 = 5$$

$$x_1 = 5 - \frac{f(5)}{f'(5)}$$

$$= 5 - \frac{8\sin\left(\frac{1}{2}\times 5\right) - 3\times 5 + 9}{4\cos\left(\frac{1}{2}\times 5\right) - 3} \quad (1)$$

$$x_1 = 4.80462$$

$$= 4.80 \text{ (3 s.f.)} \quad (1)$$

5. $$f(x) = \frac{3kx - 18}{(x+4)(x-2)} \qquad \text{where } k \text{ is a positive constant}$$

(a) Express $f(x)$ in partial fractions in terms of k.

(3)

(b) Hence find the exact value of k for which

$$\int_{-3}^{1} f(x)\,dx = 21$$

(4)

a) $f(x) = \frac{3kx-18}{(x+4)(x-2)} = \frac{A}{x+4} + \frac{B}{x-2}$

$3kx - 18 = A(x-2) + B(x+4)$ 1

$= Ax - 2A + Bx + 4B$

$= (A+B)x + (4B-2A)$

x: $3k = A + B \Rightarrow B = 3k - A$ ①

constant: $-18 = -2A + 4B$ ②

sub ① into ②: $-18 = -2A + 4(3k - A)$ 1

$-18 = -2A + 12k - 4A$

$6A = 12k + 18$

$A = 2k + 3$

$B = 3k - (2k+3) = k - 3$

$f(x) = \frac{2k+3}{x+4} + \frac{k-3}{x-2}$ 1

b) $\int_{-3}^{1} f(x)\,dx = 21 \quad \therefore \int_{-3}^{1} \frac{2k+3}{x+4} + \frac{k-3}{x-2}\,dx = 21$

$$\Big[(2k+3)\ln|x+4| + (k-3)\ln|x-2|\Big]_{-3}^{1} = 21$$

1 1

$$(2k+3)\ln 5 + \cancel{(k-3)\ln 1}^{\,0} - \cancel{(2k+3)\ln 1}^{\,0} - (k-3)\ln 5 = 21$$

$\ln 1 = 0$

$$(2k+3-k+3)\ln 5 = 21 \quad 1$$

$$(k+6)\ln 5 = 21$$

$$k+6 = \frac{21}{\ln 5}$$

$$k = \frac{21}{\ln 5} - 6 \quad 1$$

6. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

(a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$(3+x)^{-2}$$

writing each term in simplest form.

(4)

(b) Using the answer to part (a) and using algebraic integration, estimate the value of

$$\int_{0.2}^{0.4} \frac{6x}{(3+x)^2}\,\mathrm{d}x$$

giving your answer to 4 significant figures.

(4)

(c) Find, using algebraic integration, the exact value of

$$\int_{0.2}^{0.4} \frac{6x}{(3+x)^2}\,\mathrm{d}x$$

giving your answer in the form $a\ln b + c$, where a, b and c are constants to be found.

(5)

a) $(3+x)^{-2} = \left(3\left(1+\frac{x}{3}\right)\right)^{-2} = 3^{-2}\left(1+\frac{x}{3}\right)^{-2}$

$= \frac{1}{9}\left(1 - \frac{2x}{3} + \frac{(-2)(-2-1)}{2!}\left(\frac{x}{3}\right)^2 + \dots\right)$ 1 1 1

$= \frac{1}{9} - \frac{2x}{27} + \frac{x^2}{27}$ 1

b) $\int 6x(3+x)^{-2}\,dx \approx \int 6x\left(\frac{1}{9} - \frac{2x}{27} - \frac{x^2}{27}\right)dx$

$= \int\left(\frac{2x}{3} - \frac{4x^2}{9} + \frac{2x^3}{9}\right)dx$ 1

$$\int_{0.2}^{0.4}\left(\frac{2x}{3}-\frac{4x^2}{9}+\frac{2x^3}{9}\right)dx=\left[\frac{x^2}{3}-\frac{4x^3}{27}+\frac{x^4}{18}\right]_{0.2}^{0.4} \quad (1)$$

$$=\left(\frac{(0.4)^2}{3}-\frac{4(0.4)^3}{27}+\frac{(0.4)^4}{18}\right)-\left(\frac{(0.2)^2}{3}-\frac{4(0.2)^3}{27}+\frac{(0.2)^4}{18}\right) \quad (1)$$

$$=\frac{223}{6570}$$

$$=0.03304\ (4sf) \quad (1)$$

c) $\displaystyle\int_{0.2}^{0.4}\frac{6x}{(3+x)^2}\,dx$ let $u=3+x \Rightarrow x=u-3$

$du=dx$

limits: $(0.2, 0.4)\to(3.2, 3.4)$

$$=\int_{3.2}^{3.4}\frac{6(u-3)}{u^2}\,du=\int_{3.2}^{3.4}\left(\frac{6}{u}-18u^{-2}\right)du \quad (1)$$

$$=\left[6\ln|u|+18u^{-1}\right]_{3.2}^{3.4} \quad (1)\ (1)$$

$$=\left(6\ln 3.4+\frac{18}{3.4}\right)-\left(6\ln 3.2+\frac{18}{3.2}\right) \quad (1) \quad =6\ln\left(\frac{17}{16}\right)-\frac{45}{136} \quad (1)$$